JOURNAL OF APPROXIMATION THEORY 6, 378–386 (1972)

# Partly Alternating Families

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*Communicated by John R. Rice*

Received August 17, 1970

DEDICATED TO PROFESSOR J. L. WALSH
ON THE OCCASION OF HIS 75TH BIRTHDAY

## INTRODUCTION

In this note we study families of approximating functions such that some best approximations are characterized by alternation. For $g \in C[a, b]$, define

$$\| g \| = \sup \{ | g(x) | : a \leqslant x \leqslant b \}.$$

Let $F$ be an approximating function with parameter $A$ ranging over a space $P$, and let $F(A,.) \in C[a, b]$ for all $A \in P$. The Chebyshev problem is: Given $f \in C[a, b]$, find $A^* \in P$ minimizing $\| f - F(A,.) \|$ over $P$. The corresponding $F(A^*,.)$ is called a best approximation to $f$.

## 1. ALTERNATION

DEFINITION. $g \in C[a, b]$ is said to *alternate* $n$ times if there is a $n + 1$ point set $\{x_0, ..., x_n\}$ with $a \leqslant x_0 < \cdots < x_n \leqslant b$ such that

$$| g(x_0) | = \| g \|,$$
$$g(x_i) = (-1)^i g(x_0), \qquad i = 0, ..., n.$$

The set $\{x_0, ..., x_n\}$ is called an *alternant* of $g$.

DEFINITION. $F$ is said to have property $N$ (property $S$, property $NS$) of degree $n$ at $A$ if a necessary (sufficient, necessary and sufficient) condition for $F(A, \cdot)$ to be best to $g \in C[a, b]$ is that $g - F(A, \cdot)$ alternate $n$ times.

Rice [3, pp. 325–327] has characterized pairs $(F, P)$ such that $F$ has property

*NS* of degree $n$ at $A$ for all $A \in P$. This was later extended to cover $(F, P)$ such that $F$ has property *NS* of variable degree at all $A \in P$ [4, pp. 18–21]. We consider in this note cases such that $F$ has property *NS* of variable degree at some $A \in P$.

There are several reasons for such a study. First, to guarantee existence of best approximations by an alternating family, we may have to add various kinds of limits which do not have alternating properties, and we want a theory for the resulting family. Such families are given in Examples 1 and 7. Second, the alternating characterization property is a special case of a general characterization property called extremum characterizability [2, p. 375]; a study of partly alternating families thus aids in the study of families with partial extremum characterizability. Third, the uniqueness results for partly alternating families are useful in a study of the general uniqueness problem.

DEFINITION. $g$ has $n$ *sign changes* if there exists a set

$$\{x_0, \dots, x_n\},\ a \leqslant x_0 < \cdots < x_n \leqslant b$$

such that the $g$ is alternately $>0$ and $<0$ on the set.

DEFINITION. $F$ has *weak property Z* of degree $n$ at $A$ if there exists no $B$ such that $F(A, \cdot) - F(B, \cdot)$ has $n$ sign changes.

DEFINITION. $F$ has property $\mathcal{A}$ of degree $n$ at $A$, if for any integer $m < n$, any sequence $\{x_1, \dots, x_m\}$ with

$$a = x_0 < x_1 < \cdots < x_{m+1} = b$$

any sign $\sigma$, and any real $\epsilon$ with

$$0 < \epsilon < \min\{x_{j+1} - x_j : 0, \dots, m\},$$

there exists a $B \in P$, such that

$$\begin{aligned} &\| F(A, \cdot) - F(B, \cdot)\| < \epsilon, \\ \operatorname{sgn}(F(A, x) - F(B, x)) &= \sigma,\ a \leqslant x \leqslant x_1 - \epsilon \\ &= \sigma(-1)^j,\ x_j + \epsilon \leqslant x \leqslant x_{j+1} - \epsilon \\ &= \sigma(-1)^m,\ x_m + \epsilon \leqslant x \leqslant b. \end{aligned}$$

In case $m = 0$, the above sign condition reduces to

$$\operatorname{sgn}(F(A, \cdot) - F(B, \cdot)) = \sigma.$$

LEMMA 1. *Let $F$ have weak property $Z$ of degree $n$ at $A$. If $f - F(A, \cdot)$ alternates in sign on $x_0 < \cdots < x_n$ then*

$$\max\{|f(x_i) - F(B, x_i)| : i = 0,..., n\} \geqslant \min\{|f(x_i) - F(A, x_i)| : i = 0,..., n\}.$$

*Proof.* Suppose not, then $F(A, \cdot) - F(B, \cdot)$ can be shown to have $n$ sign changes.

COROLLARY. *Let $F$ have weak property $Z$ of degree $n$ at $A$, then $F$ has property $S$ of degree $n$ at $A$.*

*Proof.* Let $f - F(A, \cdot)$ alternate $n$ times with alternant $\{x_0, ..., x_n\}$. Apply the lemma.

LEMMA 2. *Let $F$ have property $\mathcal{A}$ of degree $n$ at $A$, then $F$ has property $N$ of degree $n$ at $A$.*

This is proven by Rice [3, pp. 18–19].

LEMMA 3. *Let $F$ have property NS of degree $n$ at $A$, then $F$ has weak property $Z$ of degree $n$ at $A$.*

*Proof.* Suppose $F(A, \cdot) - F(B, \cdot)$ has $n$ sign changes. We can construct continuous $f$ such that

$$\operatorname{sgn}(f - F(A, \cdot)) = \operatorname{sgn}(F(B, \cdot) - F(A, \cdot)),$$

$f - F(A, \cdot)$ alternates $n$ times and $\|f - F(A, \cdot)\| > \|f - F(B, \cdot)\|$.

LEMMA 4. *If $F$ has property NS of degree $n$ at $A$ then $F$ has property $\mathcal{A}$ of degree $n$ at $A$.*

This is proven by Rice [3, p. 21].

DEFINITION. $F$ is said to have *weak degree $n$* at $A$ if $F$ has weak property $Z$ and property $\mathcal{A}$ of degree $n$ at $A$.

From the four lemmas we obtain immediately

THEOREM 1. *A necessary and sufficient condition that $F$ have property NS of degree $n$ at $A$ is that $F$ have weak degree $n$ at $A$.*

The following definitions are useful in a study of uniqueness.

DEFINITION. A double zero of $g \in C[a, b]$ is a point $x$ in $(a, b)$ at which $g$ vanishes without a sign change.

DEFINITION. $F$ has *strong property* $Z$ of degree $n$ at $A$ if $F(A, \cdot) - F(B, \cdot)$ having $n$ zeros, counting double zeros twice, implies $F(A, \cdot) \equiv F(B, \cdot)$.

DEFINITION. $F$ has *strong degree* $n$ at $A$ if $f$ has strong property $Z$ and property $\mathcal{O}\!\mathcal{l}$ of degree $n$ at $A$.

## 2. EXAMPLES

EXAMPLE 1. This is taken from [2, p. 383]. Let $[a, b] = [0, 1]$, $P = [0, \infty)$,

$$F(\alpha, x) = \left(1 + \frac{1}{\alpha}\right)\Big/(1 + \alpha x) \qquad \alpha > 0$$

$$= 0 \qquad \alpha = 0$$

$F$ has strong degree 1 at all $\alpha > 0$. To guarantee existence the parameter 0 is required.

EXAMPLE 2. Let $F(\alpha, \cdot) = \alpha$ and let $P$ be a subset of the real line containing 0 as well as sequences $\{a_k\} \to 0$, $\{a_k'\} \to 0$, $a_k > 0$, $a_k' < 0$. Then $F$ has strong degree 1 at 0.

EXAMPLE 3. Let $[a, b] = [-1, 1]$ in Example 2 and add the function $|x|$ as an approximant; then $F$ has weak degree 1 at 0.

EXAMPLE 4. Let $[a, b] = [0, 1]$ and $F(A, x) = a_1 + a_2 x + a_3 x^2$. Let $P_1$ be the set of all $(a_1, a_2, a_3)$ for which $\| F(A, x) \| < 1$. Let $P_2$ be any other set of triples $(a_1, a_2, a_3)$, and let $P = P_1 \cup P_2$. Then $F$ has strong degree 3 at all $A \in P_1$.

EXAMPLE 5. Let us choose $P_2$ in Example 4 to be the set of triples $(a_1, a_2, 0)$ for which $-1 < a_1 + a_2 < 1$ and the line $y = a_1 + a_2 x$ has slope 3. Then $P_1 \cap P_2$ is empty. For any $B \in P_2$ there is an $A \in P_1$ such that $F(A, \cdot) - F(B, \cdot)$ has a sign change. Hence $F$ has no degree at any $B \in P_2$. It can be shown that $F$ is not extremum characterizable at any $B \in P_2$.

EXAMPLE 6. Let $[a, b] = [0, 1]$. Let $F(A, x) = a_1 + a_2 x$. Let $P'$ consist of all pairs $(a_1, 0)$ with $a_1 \leqslant 0$, and let $P''$ consist of all $(a_1, a_2)$ for which $a_1 + a_2 x > 0$ throughout $[a, b]$. Let $P = P' \cup P'' \cup (0, 1)$. Then $F$ has strong degree 1 at all $A \in P' \sim (0, 0)$, and strong degree 2 at all $A \in P''$. It has weak degree 1 at (0, 0) and no degree at (0, 1).

## 3. Uniqueness

LEMMA 5. *Let $F$ have weak degree $n$ at $A$. Let $A$ and $B$ be best to $f$, then $F(A, \cdot) - F(B, \cdot)$ has $n$ zeros, counting double zeros twice.*

*Proof.* $f - F(A, \cdot)$ must alternate $n$ times. Let $\{x_0, ..., x_n\}$ be an alternant of $f - F(A, \cdot)$. Assume without loss of generality that $f(x_0) - F(A, x_0) > 0$; then

$$(-1)^i (F(B, x_i) - F(A, x_i)) \geqslant 0, \qquad i = 0, ..., n.$$

By drawing a diagram it can be seen that the num ber of zeros is at least $n$, counting double zeros twice.

LEMMA 6. *Let $F$ have weak property Z of degree $n$ at $A$ and $F(A, \cdot) - F(B, \cdot)$ have $n$ zeros counting double zeros twice. There exists $f \in C[a, b]$ with $A, B$ best.*

*Proof.* Define $e = \| F(A, \cdot) - F(B, \cdot)\|/2$. Suppose first that $F(A, \cdot) - F(B, \cdot)$ has $n$ distinct zeros $z_1, ..., z_n$. Let $x'$ be a point such that $F(A, x') \neq F(B, x')$, say $F(B, x') > F(A, x')$. Let $\{x_0, ..., x_n\} = \{z_1, ..., z_n\} \cup \{x'\}$, with $x_0 < \cdots < x_n$. Let $j$ be the subscript for which $x_j = x'$
Define

$$f(x_i) = F(A, x_i) + (-1)^{i-j} e. \tag{1}$$

By construction, $|f(x_i) - F(B, x_i)| \leqslant e$, $i = 0, ..., n$. There is a continuous extension of $f$ to $[a, b]$ such that $\|f - F(A, \cdot)\| = e$, $\|f - F(B, \cdot)\| = e$. By (1), $f - F(A, \cdot)$ alternates $n$ times, and so $A$ is best, hence $B$ is also best. The other case to consider is where there are less than $n$ zeros but at least $n$ zeros when double zeros are counted twice. At a double zero $x$ of $F(A, \cdot) - F(B, \cdot)$ let $f(x) = F(A, x) - s \cdot e$, where $s$ is the sign of $F(B, \cdot) - F(A, \cdot)$ near $x$. If an endpoint $x$ is a zero let $f(x) = F(A, x) - s \cdot e$, $s$ the sign of $F(B, \cdot) - F(A, \cdot)$ near $x$. Between any two successive zeros of $F(A, \cdot) - F(B, \cdot)$, select a point $x$ and let $f(x) = F(A, x) + s \cdot e$, where $s$ is the sign of $F(B, \cdot) - F(A, \cdot)$ at $x$. If an endpoint $x$ is not a zero, define $f$ the same. It can be seen that $f - F(A, \cdot)$ alternates in sign on the points of definition with amplitude $e$. By construction, $|f(x) - F(B, x)| \leqslant e$ for such $x$. The number of points of definition of $f$ is the number of endpoint zeros plus the number of double zeros plus (1 plus the number of interior zeros, double or not), and is thus $\geqslant n + 1$. There is a continuous extension of $f$ to $[a, b]$ such that $\| f - F(A, \cdot)\| = e$, $\|f - F(B, \cdot)\| = e$. As $f - F(A, \cdot)$ alternates $n$ times, $F(A, \cdot)$ is best and so $F(B, \cdot)$ is best. From the two previous lemmas we obtain

THEOREM 2. *Let $F$ have property NS of degree $n$ at $A$. A necessary and sufficient condition for $F(A, \cdot)$ to be unique when it is best is that $F$ has strong property Z of degree $n$ at $A$.*

Strong property $Z$ is a difficult property to verify directly and we consider when it can be replaced by weaker properties, in particular weak property $Z$ on part of $P$. The following lemma is a generalization of Lemma 1 of [1], for which no complete proof was given,

LEMMA 7. *Let $F$ have weak property $Z$ of degree $n$ at $A$ and $F$ have property $\mathfrak{A}$ of degree $n$ at $B$. If $F(A, \cdot) - F(B, \cdot)$ has $n$ zeros, counting double zeros twice, but does not vanish identically, there exists $C \in P$ such that $F(A, \cdot) - F(C, \cdot)$ has $n$ sign changes.*

*Proof.* The first possibility is that $F(A, \cdot) - F(B), \cdot)$ vanishes on a nondegenerate interval $I$. Without loss of generality we can suppose that there exists $y$ such that $F(A, y) - F(B, y) > \epsilon$. By property $\mathfrak{A}$ of degree $n$ at $B$ there exists $C \in P$ such that $F(B, \cdot) - F(C, \cdot)$ changes sign $n - 1$ times in the interior of $I$,

$$\| F(B, \cdot) - F(C, \cdot)\| < \epsilon \qquad \text{and} \quad F(B, x) - F(C, x) < 0$$

for $x$ between $y$ and $I$. Then $F(A, \cdot) - F(C, \cdot)$ has a sign change between $I$ and $y$, and $n - 1$ sign changes in $I$. We have $n$ sign changes, contradicting property $Z$ of degree $n$ at $A$. The first possibility cannot occur and between any two zeros of $F(A, \cdot) - F(B, \cdot)$ there is a point at which $F(A, \cdot) - F(B, \cdot)$ does not vanish. Next suppose that $F(A, \cdot) - F(B, \cdot)$ does not change sign, say $F(A, \cdot) - F(B, \cdot) > 0$. Select a finite number of zeros $\{z_k : k = 1,..., m\}$. Between $z_k$ and $z_{k+1}$, select $x_k$ such that $F(A, x_k) - F(B, x_k) > 0$. Define

$$\epsilon = \min\{F(A, x_k) - F(B, x_k) : k = 1,..., m - 1\}.$$

By property $\mathfrak{A}$ of degree $n$ at $B$ we can select $C$ such that

$$F(C, \cdot) - F(B, \cdot) > 0, \| F(C, \cdot) - F(B, \cdot)\| < \epsilon/2.$$

It is not difficult to see that for every zero of $F(A, \cdot) - F(B, \cdot)$, counting double zeros twice, there is a sign change of $F(A, \cdot) - F(C, \cdot)$. Finally suppose that $F(A, \cdot) - F(B, \cdot)$ has exactly $k$ sign changes which occur at $z_1 ,..., z_k$ (there can be at most $n - 1$ sign changes). Select a finite set $Z$ of zeros of $F(A, \cdot) - F(B, \cdot)$ which includes $z_1 ,..., z_k$. Select a finite point set $X$ such that between any two elements of $Z$ there is an element of $X$ and

$$F(A, x) - F(B, x) \neq 0 \qquad \text{for} \quad x \in X.$$

Define

$$\epsilon_1 = \min\{| F(A, x) - F(B, x)| : x \in X\}.$$

Let $\epsilon_2 = \inf\{| x - z | : x \in X, z \in Z\}$ and set $\epsilon = 1/4 \min\{\epsilon_1, \epsilon_2\}$. By property $\mathscr{A}$ of degree $n$ at $B$ there exists $C$ such that

$$\| F(C, \cdot) - F(B, \cdot)\| < \epsilon, F(C, \cdot) - F(B, \cdot)$$

changes sign in an $\epsilon$-neighborhood of $z_i$, $i = 1,..., k$, and outside the $\epsilon$-neighbourhood, $\operatorname{sgn}(F(C, y) - F(B, y))$ is the sign of $F(A, \cdot) - F(B, \cdot)$ at or near $y$. It is not difficult to see that $F(A, \cdot) - F(C, \cdot)$ has a sign change for every zero of $F(A, \cdot) - F(B, \cdot)$, counting double zeros twice.
In case $(F, P)$ is an alternating family, all elements have a degree (as defined in ref. 1, p. 225) and we have

COROLLARY. *Let $F$ have degree $n$ at $A$ and some degree at $B$. If $F(A, \cdot) - F(B, \cdot)$ has $n$ zeros, counting double zeros twice, then*

$$F(A, \cdot) \equiv F(B, \cdot).$$

*Proof.* We go through the same arguments as in the proof of the lemma. $F(A, \cdot) - F(B, \cdot)$ cannot vanish on an interval without vanishing everywhere. In case $F(A, \cdot) - F(B, \cdot)$ is not identically zero and does not change sign, there exists $C \in P$ such that $F(A, \cdot) - F(C, \cdot)$ has $n$ sign changes, contrary to hypothesis. In case $F(A, \cdot) - F(B, \cdot)$ has $k$ sign changes, there must be at least one other zero of $F(A, \cdot) - F(B, \cdot)$, hence the degree of $F$ at $B$ must be at least $k + 2$. As $F$ has property $\mathscr{A}$ of degree $k + 2$ at $B$, there exists $C \in P$ such that $F(A, \cdot) - F(C, \cdot)$ has $n$ sign changes, contrary to hypothesis.

THEOREM 3. *Let $Q$ be the set of elements of $P$ at which $F$ has a weak degree. Let $F$ have weak degree $n$ at $A$. If $F(A, \cdot)$ is a best approximation to $f$ there is no other best approximation in $Q$. A necessary and sufficient condition for $F(A, \cdot)$ to be unique whenever it is best is that for all $B \in P \sim Q$, $F(A, \cdot) - F(B, \cdot)$ has less than $n$ zeros, counting double zeros twice.*

*Proof.* Suppose $A$, $B$ are best and in $Q$. Let $F$ have weak degree $m$ at $B$. Assume without loss of generality that $m \geqslant n$. $F$ has property $\mathscr{A}$ of degree $m$ at $B$, hence property $\mathscr{A}$ of degree $n$ at $B$. By Lemma 5, $F(A, \cdot) - F(B, \cdot)$ has $n$ zeros, counting double zeros twice and by Lemma 7, $F(A, \cdot) \equiv F(B, \cdot)$. It follows that if $F(A, \cdot)$ is best and $F(B, \cdot)$ is a different best approximation, $B \in P \sim Q$. By lemma 5, $F(A, \cdot) - F(B, \cdot)$ has $n$ zeros counting double zeros twice, establishing sufficiency. If $F(A, \cdot) - F(B, \cdot)$ has $n$ zeros, counting double zeros twice, Lemma 6 guarantees nonuniqueness, proving necessity.

Let us consider Example 4 of Section 2. $F$ has strong property $Z$ of degree 3 at all elements. By the above theorem nonuniqueness at best approximations is possible only if two best parameters exist in $P_2$. If we select $P_2$ so that best approximations from parameter space $P_2$ are unique, we have uniqueness for all continuous functions. In particular, we have uniqueness for Example 5, even though $F$ has neither degree nor extremum characterizability at any parameter in $P_2$.

## 4. Noncontinuous Approximations

To get existence of best approximations, we may have to add noncontinuous limits to an alternating family.

Example 7. Let

$$F(\alpha, x) = 1/(1 + \alpha x) + (1/\alpha), \qquad x < 0$$
$$= 1 + (1/\alpha), \qquad x \geqslant 0$$

for $\alpha > 0$. Let $P = (1, \infty)$. If $\alpha_1 < \alpha_2$ then $F(\alpha_1, x) < F(\alpha_2, x)$, so $F$ has strong property $Z$ of degree 1 at all $\alpha \in P$. Using Dini's theorem we can show that $F$ has property $\mathcal{A}$ of degree 1 at all $\alpha \in P$. However, best approximations do not exist to all continuous functions. To assure such existence we must add $F(1, \cdot)$, which is continuous, and $F(\infty, \cdot)$, which is not continous, as approximations.

The theory obtained previously is valid for noncontinuous approximants providing $F$ having a weak degree at $A$ and $F(B, \cdot)$ being noncontinuous imply that $F(A, \cdot) - F(B, \cdot)$ has no sign changes or zeros.

## 5. Computation of Best Approximations

An algorithm (a variant of the Remez algorithm) for computing best approximations by alternating families is described in [1, p. 228 ff.]. This algorithm can be used to compute best approximations by partly alternating families. Suppose there is a unique parameter $A^*$ that is best and suppose $F$ has strong degree $n$ (the maximum degree) at all $A$ in a neighborhood of $A^*$. If the hypotheses of Theorem 2 of [1, p. 229] are satisfied, the algorithm has quadratic convergence.

## References

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